Exercise 1

Show that the Cauchy problem for the Laplace equation:

$$\begin{cases} v_{tt} + v_{xx} = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ v|_{t=0} = f(x), & v_t|_{t=0} = g(x) \end{cases}$$

is ill-posed in $C^k(\mathbb{R})$, $\forall k \in \mathbb{N}$.
Hint: Show that the above problem with the initial data sequence

$$(v_n, (v_n)_t)|_{t=0} = (0, g_n) = (0, e^{-\sqrt{n}} n \sin(nx))$$

have a unique solution $v_n(t,x)$. However for any positive time $t_0 > 0$,

$$|v_n(t_0, x)| = |e^{-\sqrt{n}} \sin(nx) \mathrm{sh}(nt_0)| \to \infty \quad \text{in} \quad C^k(\mathbb{R}).$$

Exercise 2

Recall the definition of the Schrödinger group $S(t)$:

$$S(t)g = e^{it\Delta} g = K_t * g = (4\pi i t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} g(y) dy. \tag{St}$$

Show that the operator $S(t)$, $t > 0$ does not map

- from $L^2(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ or from $L^r(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ if $r \neq 2$;
- from $L^r(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ if $r \neq 2$;
- from $L^r(\mathbb{R}^d)$ to $L^{r_1}(\mathbb{R}^d)$ for any $r > 2$;
- from $H^s(\mathbb{R}^d)$ to $H^{s'}(\mathbb{R}^d)$ for $s' > s$.

Exercise 3

Show the Hardy-Littlewood-Sobolev (generalized Young's) inequality

$$\|g * |\cdot|^{-\alpha}\|_{L^q(\mathbb{R}^n)} \leq C(m,q,\alpha,n) \|g\|_{L^m(\mathbb{R}^n)},$$

where $1 + \frac{1}{q} = \frac{1}{m} + \frac{\alpha}{n}$, $\quad 0 < \alpha < n$ and $\quad 1 < m < q < \infty$.
Hint: Follow the following steps:
Step1. Decompose the convolution in the following way

$$\left(g * |\cdot|^{-\alpha}\right)(x) = \underbrace{\int_{|x-y|\leq R} \frac{g(y)}{|x-y|^\alpha}\, dy}_{(1)} + \underbrace{\int_{|x-y|>R} \frac{g(y)}{|x-y|^\alpha}\, dy}_{(2)}.$$

Step2. Control the term (1) by the centered Hardy-Littlewood maximal function up to a constant which depends on R. The centered Hardy-Littlewood maximal function of f is defined as

$$\begin{aligned}\mathcal{M}f(x) &= \sup_{r>0} \fint_{B_r(x)} |f| \\ &= \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f|.\end{aligned}$$

Step3. Control the term (2) by $\|g\|_{L^m(\mathbb{R}^n)}$ up to a constant which depends on R.
Step4. Compare the term (1) and the term (2) to choose proper R.

Exercise 4

Take $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\varphi \geq 0$ and $\int_{\mathbb{R}^d} \varphi(x) = 1$. Denote $\varphi_n(x) = n^d \varphi(nx)$. If $u : \mathbb{R}^d \to \mathbb{R}$ is locally integrable, we define its mollification

$$u^n = \varphi_n * u.$$

Show that:

- If $u \in L^p(\mathbb{R}^d)$ with $1 \leq p < \infty$, then $u^n \to u$ in $L^p(\mathbb{R}^d)$.
- If $u \in C(\mathbb{R}^d)$, then $u^n \to u$ uniformly on compact subsets of $\mathbb{R}^d$.